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# Gravitation of fast-moving particles

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**Abstract.** Melissinos has recently discussed two possible experiments on the gravitation of a bunch of charged particles moving in a storage ring. In one experiment the gravitational field of the bunch is measured by a detector, and in the second the bunch is deflected from its circular path by a large mass outside the ring. In this paper the theoretical basis of the experiments is examined by using the properties of Schwarzschild space–time.

## 1. Introduction

In an interesting paper Melissinos (1981) has studied gravitation of fast-moving charged particles circulating in a bunch round a storage ring. He has considered two possible experiments: first, to use the bunch as a source of gravitation and to measure its effect on a detector; secondly, to measure the deflection of the bunch from its circular path by the field of a large static mass. In setting up a theoretical basis for these experiments Melissinos makes two approximations. He approximates the circular form of the storage ring by an infinite straight line; and he uses the linear approximation to Einstein's vacuum equations in harmonic coordinates.

From the standpoint of general relativity it is fairly easy to construct a simple model of the second experiment. In this paper the bunch will be considered as a charged test particle moving in an applied magnetic field and in the gravitational field of a large mass; radiation, gravitational and electromagnetic, will be neglected. One can then calculate the motion of the bunch by the geodesic equation, augmented by the electromagnetic force term, in the Schwarzschild space–time. Proceeding thus, one need not approximate the orbit by a straight line, nor does one need the linear approximation in its usual form. This calculation is done in § 2. The result is quite different from that of Melissinos and seems to be well below the limit of experimental detection. How far the discrepancy is due to the model being an oversimplification is discussed in the conclusion, § 4.

No satisfactory existing theoretical framework for the first experiment is known to me. Uniform straight line motion of the bunch is adequately represented by the linear approximation to Einstein's vacuum equations; indeed, it is the only motion which this approximation *can* legitimately represent (Weyl 1944, Havas and Goldberg 1962). However, the actual bunch moves in a circle, and to do this it requires the intervention of a non-gravitational agency, namely a magnetic field. The gravitational field of the latter may not be neglected if one wishes a rational, consistent theory of the experiment.

  

To obtain a solution of Einstein's equations representing the gravitational field of the bunch plus that of the necessary applied magnetic field would be, even in approximation, very difficult. In this paper (§ 3) I content myself with studying the straight line motion, but from a point of view different from that of Melissinos. I submit the Schwarzschild solution to a Lorentz transformation and examine the geodesic motion of a test mass. My result here agrees with that of Melissinos, except for an extra term in the equation of motion.

The paper contains two appendices. In the first the coordinate components of the magnetic field are derived by solving Maxwell's equations on the background of the Schwarzschild metric due to the large mass. In appendix 2 I give the mathematics necessary to derive the equation of motion (3.9).

## 2. Deflection of the beam by a massive body

We study the effect of the gravitational field of a large mass $M$ on a bunch of particles moving round the storage ring. We idealise the situation as follows. The bunch is considered as a single charged test mass, moving nearly in a circle under an applied magnetic field orthogonal to the ring, and the gravitational field of $M$. We ignore auxiliary fields which are used in practice and we consider the magnetic field to be uniform, subject to small corrections due to general relativity. $M$ is supposed to lie in the plane of the ring, and outside it. (Much the same analysis applies if $M$ is inside.)

The gravitational field of $M$ is given by the Schwarzschild metric, and it will be sufficient to use the linearised form, i.e. to ignore powers of $M$ higher than the first:

$$ds^2 = -(1+2\mu r^{-1})\,dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + c^2(1-2\mu r^{-1})\,dt^2, \tag{2.1}$$

where

$$\mu = GMc^{-2}, \tag{2.2}$$

$G$ being the gravitational constant and $c$ the speed of light. We shall number the coordinates as follows:

$$x^1 \equiv r, \qquad x^2 \equiv \theta, \qquad x^3 \equiv \phi, \qquad x^4 \equiv t. \tag{2.3}$$

$M$ is taken at the origin O of polar coordinates, and the storage ring, of coordinate radius $a$, has its centre A at $r = a + b$, $\theta = \frac{1}{2}\pi$, $\phi = 0$, where

$$a \gg b \gg 2\mu > 0. \tag{2.4}$$

The ring lies in the plane $\theta = \frac{1}{2}\pi$ (see figure 1).

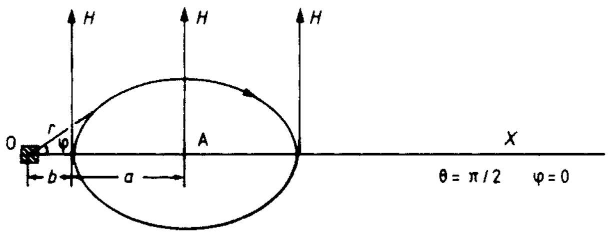


**Figure 1.** The mass $M$ is at O and the bunch moves in a circle, in the plane $\theta = \frac{1}{2}\pi$, under the influence of the magnetic field $H$.

The applied magnetic field is static, asymptotically uniform, and orthogonal to the ring. It must satisfy Maxwell's equations on the background of the approximate Schwarzschild metric (2.1). These equations are solved in appendix 1, and it is shown that the only non-vanishing components of the electromagnetic field tensor at a point in the plane $\theta = \frac{1}{2}\pi$ are

$$-F^{31} = F^{13} = r^{-1}H(1-2\mu r^{-1}), \tag{2.5}$$

where $H$ is a constant. This corresponds to a uniform magnetic field of strength $H$, but with a correcting term due to the gravitational field of $M$.

The motion of the bunch is given by the geodesic equation augmented by the term expressing the electromagnetic force:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}s^2} + \Gamma^i_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s} = \frac{e}{m_0c^2}F^{ai}g_{ab}\frac{\mathrm{d}x^b}{\mathrm{d}s}, \tag{2.6}$$

$e$ and $m_0$ being the charge and proper mass of the bunch, and $\Gamma^i_{ab}$ the Christoffel symbols for (2.1); radiation terms are neglected. Gaussian units are used for electromagnetic quantities. The motion takes place in $\theta = \frac{1}{2}\pi$, and is governed by three equations, namely (2.1), and (2.6) with $i = 3$ and 4. These may be written

$$\left(1+\frac{2\mu}{r}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 - c^2\left(1-\frac{2\mu}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 = -1, \tag{2.7}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = -\frac{\omega}{r}\frac{\mathrm{d}r}{\mathrm{d}s}, \tag{2.8}$$

$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2} + \frac{2\mu}{r^2}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}t}{\mathrm{d}s} = 0, \tag{2.9}$$

once again ignoring terms in $M^2$, and putting

$$\omega = eH/mc^2. \tag{2.10}$$

We integrate (2.8) and (2.9) to get

$$r^2\,\mathrm{d}\phi/\mathrm{d}s = h - \tfrac{1}{2}\omega r^2, \tag{2.11}$$

$$\mathrm{d}t/\mathrm{d}s = k(1+2\mu/r), \tag{2.12}$$

$h$ and $k$ being constants of integration. Substituting these into (2.7) we obtain after simplification

$$\begin{aligned}(\mathrm{d}r/\mathrm{d}s)^2 &= r^{-3}[-\tfrac{1}{4}\omega^2r^5 + \tfrac{1}{2}\omega^2\mu r^4 + (\omega h + c^2k^2 - 1)r^3 + 2\mu r^2(1-\omega h) - h^2r + 2\mu h^2],\\ &=: r^{-3}Q,\end{aligned} \tag{2.13}$$

introducing $Q$ to denote the expression in the square bracket.

If $\mu = 0$, equations (2.11)–(2.13) are special relativistic equations referring to the circular motion of a charged particle in a uniform magnetic field, determined by the constants of the motion $h$ and $k$ related to the angular momentum and the energy. We can express these constants in terms of the distances $a$ and $b$ by noting that $\mathrm{d}r/\mathrm{d}s = 0$ when $r = b$ and when $r = 2a + b$, which gives from (2.13) with $\mu = 0$

$$h = \tfrac{1}{2}\omega b(2a+b), \tag{2.14}$$

$$c^2k^2 = 1 + \omega^2a^2. \tag{2.15}$$

Finally we recall that the period of the circular motion when $M = 0$ is $2\pi(\omega c)^{-1}(1 + \omega^2 a^2)^{1/2}$, a result best obtained by using A as origin and solving (2.6).

We shall now allow the constants $h$ and $k$ in (2.13) to retain their values (2.14) and (2.15), but let $M$ be non-zero, while assuming (2.4). This means physically that we are studying a motion, with the same energy and angular momentum as before, but with the mass $M$ present at O. The new motion must satisfy (2.13) with $\mu \neq 0$.

To find the effect of $M$ on the orbit we study the roots of $dr/ds = 0$,or of

$$Q = 0. \tag{2.16}$$

When $M = 0$ the roots are, of course,

$$r = \pm b, \qquad r = \pm(2a + b) \tag{2.17}$$

(the root $r = 0$ of (2.16) with $M = 0$ has no physical significance because of the factor $r^{-3}$ in (2.13)). This corresponds to the closed circular orbit referred to above. The effect of $M \neq 0$ is to make small changes in these roots, as is seen in figure 2 in which the broken and unbroken lines refer to cases $M = 0$ and $M \neq 0$ respectively. An additional positive root, shown at O′, occurs at a distance from O of the order of the Schwarzschild radius and is of no physical relevance here. Thus when the mass is present the bunch moves periodically between radial coordinate distances OB′ and OC′. The orbit is slightly different from the circular one obtained when $M = 0$, but *the point closest to the position of M, namely B′ in figure 2, does not drift towards M in successive revolutions, as suggested by Melissinos's analysis.*

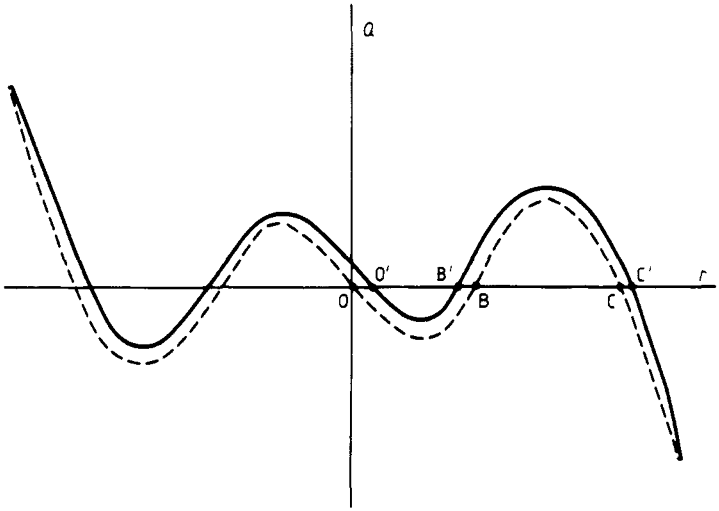


**Figure 2.** The plot of $Q$ against $r$; the broken and unbroken curves refer to the cases of $M = 0$ and $M \neq 0$ respectively.

The intercept of the orbit on $\phi = 0$, which I shall call the major axis, is given by B′C′ in figure 2. We can calculate this by finding the small corrections to the roots (2.17) of (2.16). Confining attention to positive roots, we substitute

$$r = b - \varepsilon$$

into (2.16), neglect powers of $\varepsilon$ higher than the first, and find, using (2.2),

$$\varepsilon = \mathrm{B'B} = aGMc^{-2}(a + b)^{-1}[1 + (a\omega)^{-2}]. \tag{2.18}$$

Similarly, substituting

$$r = 2a + b + \eta$$

and neglecting powers of $\eta$ higher than the first, we obtain

$$\eta = CC' = aGMc^{-2}(a+b)^{-1}[1+(a\omega)^{-2}],$$

which is equal to $\varepsilon$ as given by (2.18). *The effect of the mass $M$ is therefore to lengthen the major axis of the orbit by* $2aGMc^{-2}(a+b)^{-1}[1+(a\omega)^{-2}]$ *and to bring the perihelion nearer to $M$ by a distance $\varepsilon$ given by* (2.18).

Choosing numbers similar to those of Melissinos (1981), we take

$$M = 3000\ \text{kg}, \qquad a = 10^5\ \text{cm}, \qquad b = 10\ \text{cm}, \qquad \omega = 10^{-5}\ \text{cm}^{-1}, \tag{2.19}$$

and find

$$\varepsilon \sim 4 \times 10^{-22}\ \text{cm},$$

*which seems quite undetectable.*

One can, of course, solve the same problem using only Newtonian mechanics and the Lorentz force; the answer is

$$\varepsilon_N = GMc^{-2}[\omega^2 a(a+b)]^{-1}.$$

Therefore the relativistic theory simply gives an extra term

$$aGMc^{-2}(a+b)^{-1}.$$

These terms are approximately equal if one uses the data (2.19).

## 3. Response of a mass to the moving bunch

In this case we have to find the acceleration of a test mass in the gravitational field of the bunch of charged particles moving in a circle round the ring. Melissinos (1981), using the linear approximation to Einstein's vacuum equations, considers instead the gravitational effect of the bunch when moving in an infinite straight line (figure 3), supposing that this will approximate the circular motion if the distance of closest approach is the same in both cases.

From the staindpoint of general relativity these two motions are entirely different, and it is by no means clear that they give comparable accelerations to the test mass. The straight line motion requires no external forces, and indeed its field is simply a Lorentz transformation of the Schwarzschild solution. The circular motion requires an external agency (namely, the magnetic field) which will contribute to the gravitational field and ought to be taken into account. *The linear approximation to Einstein's vacuum equations is incapable of giving a solution to the problem of circular motion* because the only motion compatible with it is that of constant velocity in a straight line (Weyl 1944, Havas and Goldberg 1962). Thus Melissinos's method is adequate for the problem he discusses but not for the one he is really interested in.

For straight line motion, Melissinos's result, altered in one respect, can be simply obtained by submitting the linearised Schwarzschild solution to a Lorentz transformation. We first replace $\mu$ in (2.1) by

$$\lambda = Gmc^{-2} \tag{3.1}$$

where $m$ is the active gravitational mass of the bunch. Next we put (2.1) thus revised into isotropic form

$$ds = -(1+2\lambda R^{-1})(dx^2+dy^2+dz^2)+c^2(1-2\lambda R^{-1})\,dt^2, \qquad R^2 = x^2+y^2+z^2, \tag{3.2}$$

by means of the transformation

$$x=(r-\lambda)\sin\theta\cos\phi, \qquad y=(r-\lambda)\sin\theta\sin\phi, \qquad z=(r-\lambda)\cos\theta,$$

and neglect $m^2$. We now submit (3.2) to the Lorentz transformation

$$x=\gamma(X+vT), \qquad y=Y, \qquad z=Z, \qquad t=\gamma(T+vXc^{-2}), \tag{3.3}$$

where $\gamma=(1-v^2c^{-2})^{-1/2}$, so that

$$R^2=\gamma^2(X+vT)^2+Y^2+Z^2. \tag{3.4}$$

The bunch is now moving with velocity $-v$ along the $X$ axis and the metric is

$$ds^2=-\left(1+\frac{2\lambda\delta}{R}\right)dX^2-\left(1+\frac{2\lambda}{R}\right)(dY^2+dZ^2)+c^2\left(1-\frac{2\lambda\delta}{R}\right)dT^2-\frac{8v\gamma^2\lambda}{R}\,dX\,dT, \tag{3.5}$$

where $\delta=(c^2+v^2)(c^2-v^2)^{-1}$. This agrees with Melissinos except that he omits the term $2\lambda/R$ in the coefficient of $(dY^2+dZ^2)$.

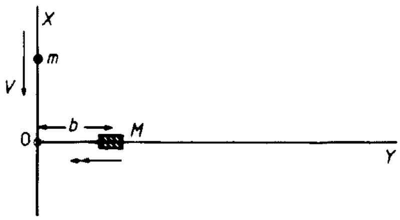


**Figure 3.** The gravitational effect of the bunch, mass $m$, on the detecting mass $M$. $m$ is moving in the negative $X$ direction.

Melissionos's procedure now is to calculate the acceleration of the mass $M$ in the gravitational field of the bunch by using the geodesic equations on (3.5). $M$ is assumed to be constrained on the $Y$ axis (figure 3) so its coordinates are

$$(0,\ Y,\ 0), \tag{3.6}$$

and its distance from the bunch $m$ is, by (3.4),

$$R_M=+(Y^2+v^2\gamma^2T^2)^{1/2}. \tag{3.7}$$

The gravitational acceleration of $M$ is given by the geodesic equation

$$\frac{d^2Y}{ds^2}+\Gamma^2_{22}\left(\frac{dY}{ds}\right)^2+2\Gamma^2_{24}\frac{dY}{ds}\frac{dT}{ds}+\Gamma^2_{44}\left(\frac{dT}{ds}\right)^2=0, \tag{3.8}$$

indices 2 and 4 referring to $Y$ and $T$ respectively. For small, slow oscillations of the mass $M$ we may assume $dY/ds \ll 1$, and $ds=c\,dT$ is an adequate approximation (appendix 2). Calculating the $\Gamma$s from (3.5) we find, as shown in appendix 2,

$$\frac{d^2Y}{dT^2}+\frac{mG}{R_M^3}\left(Y\delta-\frac{(\delta+2)\gamma^2v^2T}{c^2}\frac{dY}{dT}\right)=0 \tag{3.9}$$

where we have neglected terms of order $\lambda^2$, and of order $\lambda(\mathrm{d}Y/c\,\mathrm{d}T)^2$, but retained the term in $\mathrm{d}Y/\mathrm{d}T$ shown. Supposing that $v$ is comparable to $c$, we find for the modulus of the ratio of the second term in the large brackets to the first

$$\frac{(\delta+2)\gamma^2 v^2 T}{c^2 Y\delta}\frac{\mathrm{d}Y}{\mathrm{d}T}\sim\frac{\gamma^2 T}{Y}\frac{\mathrm{d}Y}{\mathrm{d}T}\sim 10^{-6}\gamma^2\frac{\mathrm{d}Y}{\mathrm{d}T},$$

taking $T\sim 10^{-5}$ s (the period of the circular motion) and $Y\sim b=10$ cm. This ratio is not necessarily small. The term in $\mathrm{d}Y/\mathrm{d}T$ damps the motion for $T<0$ and anti-damps for $T>0$. Neglecting this term in (3.9), we have

$$\mathrm{d}^2Y/\mathrm{d}T^2=-mGY\delta/R_M^3, \tag{3.10}$$

which is the result found by Melissinos. It represents Newtonian inverse square law attraction multiplied by the factor $\delta$, which can be large.

## 4. Conclusion

In constructing a theoretical model for Melissinos's proposed first experiment, one meets the difficulties mentioned in the Introduction. Evading these by the questionable device of studying straight-line instead of circular motion, I found (3.9) for the equation of motion of the detecting mass in the gravitational field of the bunch, agreeing with Melissinos's result except for one term. If this procedure is indeed admissible the result shows that there can be considerable enhancement of Newtonian gravitational attraction.

Studying Melissinos's second experiment, in § 2, I obtained a result different from his, and below the limit of experimental detection. However, my model is an over-simplification. The charged particles in a bunch moving at the velocity contemplated would radiate electromagnetic waves very strongly—at a rate sufficient to lose their entire rest mass in less than one revolution. In a storage ring this energy would be returned to the particles by means of the restoring magnetic forces specially provided to stabilise the beam, and distinct from the uniform magnetic field appearing in my model. To take account of this one would need to include the restoring forces in the term on the right of (2.6), as well as adding to that equation a term corresponding to the radiation loss. It is totally obscure how this would affect the result I found.

Melissinos's analysis of the second experiment is quite different from mine. He supposes that every time it passes the static mass the bunch receives an impulse, and the cumulative effect of these impulses is a drift in the orbit which eventually becomes measurable. This treatment seems to take no account of the restoring effect of the primary and secondary magnetic fields.

This question is still quite open, and deserves further attention.

## Appendix 1. Asymptotically uniform magnetic field on a Schwarzschild background

We seek a static solution of Maxwell's equations in free space

$$F_{[ik,l]}=0, \qquad \frac{\partial}{\partial x^i}(\sqrt{-g}F^{ik})=0, \tag{A1.1}$$

corresponding to a uniform magnetic field on the approximate Schwarzschild metric (2.1). Here $g$ is the determinant of (2.1).

Numbering the coordinates as in (2.3), we can satisfy the first of (A1.1) by choosing a vector potential

$$A_i = \delta_i^3 P(r) \sin^2 \theta,$$

determining $F_{ik}$ by

$$F_{ik} = A_{i,k} - A_{k,i},$$

the comma meaning partial differentiation, and $P$ being a function to be determined. Raising indices by means of (2.1), we find that the second of (A1.1) demands that $P$ satisfy the equation

$$\left(1-\frac{2\mu}{r}\right)P'' + \frac{2\mu}{r^2}P' - \frac{2P}{r^2} = 0 \tag{A1.2}$$

where $' = \mathrm{d}/\mathrm{d}r$. The solution of (A1.2) is

$$P = -\tfrac{1}{2}Hr^2 + L[r^2 \log(1-2\mu/r) + 2\mu r + 2\mu^2],$$

$H$ and $L$ being arbitrary constants. The coefficient of $L$ does not refer to an asymptotically uniform field, so we put $L=0$ and find

$$F_{31} = -Hr\sin^2\theta, \qquad F_{32} = -Hr^2 \sin\theta\cos\theta.$$

On $\theta = \frac{1}{2}\pi$ we have $F_{31} = -Hr$, $F_{32} = 0$ and on raising indices by means of (2.1) we find (2.5).

## Appendix 2. Derivation of equation of motion (3.9)

The object is to derive (3.9) from the geodesic equation (3.8). The Christoffel symbols, obtained from the metric (3.5), are

$$\Gamma^2_{22} = -\frac{\lambda Y}{R^3}, \qquad \Gamma^2_{24} = -\frac{\lambda\gamma^2 v^2 T}{R^3}, \qquad \Gamma^2_{44} = \frac{\lambda Y \delta c^2}{R^3}, \tag{A2.1}$$

terms of order $\lambda^2$ being neglected. To deal with the terms $\mathrm{d}Y/\mathrm{d}s$ and $\mathrm{d}T/\mathrm{d}s$ in (3.8) we start from the metric (3.5), remembering that in the displacement of interest $\mathrm{d}X = \mathrm{d}Z = 0$; so

$$\mathrm{d}s^2 = -(1+2\lambda/R)\,\mathrm{d}Y^2 + c^2(1-2\lambda\delta/R)\,\mathrm{d}T^2,$$

whence we find

$$\frac{\mathrm{d}T}{\mathrm{d}s} = \frac{1}{c}\left[1 - \frac{2\lambda\delta}{R} - \left(1+\frac{2\lambda}{R}\right)\left(\frac{1}{c}\frac{\mathrm{d}Y}{\mathrm{d}T}\right)^2\right]^{-1/2}. \tag{A2.2}$$

Thus, neglecting terms of order $\lambda$ and of order $(\mathrm{d}Y/c\,\mathrm{d}T)^2$, we have

$$c\,\mathrm{d}T = \mathrm{d}s. \tag{A2.3}$$

We need to replace $\mathrm{d}^2Y/\mathrm{d}s^2$ in (3.8) by

$$\frac{\mathrm{d}^2Y}{\mathrm{d}s^2} = \frac{\mathrm{d}^2Y}{\mathrm{d}T^2}\left(\frac{\mathrm{d}T}{\mathrm{d}s}\right)^2 + \frac{\mathrm{d}Y}{\mathrm{d}T}\frac{\mathrm{d}^2T}{\mathrm{d}s^2}. \tag{A2.4}$$

$d^2T/ds^2$ is obtained by differentiating (A2.2), or by use of the geodesic equation in $T$ corresponding to (3.8), and we find

$$d^2T/ds^2 = -\lambda v^2\gamma^2\delta T/c^2R^3, \tag{A2.5}$$

neglecting terms of orders $\lambda^2$, $\lambda\, dY/c^2\, dT$. Substituting (A2.1), (A2.4) and (A2.5) into (3.8), and using (A2.3), we obtain (3.9).

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